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Decay of the Bethe–Salpeter Kernel and Bound States for Lattice Classical Ferromagnetic Spin Systems at High Temperature

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We consider lattice classical ferromagnetic spin systems at high temperature ($\beta \ll 1$) with nearest neighbor interactions and even single-spin distributions (ssd). Associated with each system is an imaginary time lattice quantum field theory. It is known that there is a particle of mass $m \cong -\ln \beta$ in the energy-momentum spectrum. If $\alpha \equiv \langle s^4 \rangle - 3\langle s^2 \rangle^2 < 0$, where $\langle s^k \rangle$ is the kth moment of the ssd, and β is sufficiently small, we show that in the two-particle subspace there is no mass spectrum up to $2m$. For $\alpha > 0$ we show that the only mass spectrum in $(m, 2m)$ is a bound state of mass $m_b = 2m + \ln(1-\gamma) + O(\beta)$, where $\gamma = \alpha(\alpha + 2\langle s^2 \rangle^2)^{-1}$. A bound on the decay of the kernel of a Bethe–Salpeter equation is obtained and used to prove these results.

KEY WORDS: Transfer matrix spectrum; decay of correlations; bound states; high-temperature ferromagnetic spin systems; Gaussian domination inequalities.

1. INTRODUCTION AND RESULTS

Consider the lattice imaginary time quantum field theory (qft) associated with lattice classical ferromagnetic spin systems at high temperature ($\beta \ll 1$) (see refs. 1 and 2). We only treat spin systems with nearest neighbor interactions and even single spin distributions (ssd). It is known that the energy-momentum (e-m) spectrum has a particle of mass $m \sim -\ln \beta$ with an isolated dispersion curve (see refs. 3 and 4). Recently using a Bethe–Salpeter (B–S) equation we have shown in refs. 5 and 6 that

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a bound state exists if $\alpha \equiv \langle s^4 \rangle - 3\langle s^2 \rangle^2 > 0$ where $\langle s^k \rangle$ is the kth moment of the ssd. The bound state mass is given by $m_b = 2m + \ln(1-\gamma) + O(\beta)$ where $\gamma = \alpha(\alpha + 2\langle s^2 \rangle^2)^{-1}$. A crucial input is the decay of the kernel of the B–S equation. In this article we provide the proof of the bounds on the kernel using a decoupling of hyperplane method (see ref. 4). In addition if $\alpha < 0$ and $\langle s^2 \rangle^2 \neq \langle s^4 \rangle$ we show that in the two-particle subspace there is no mass spectrum up to the two-particle threshold $2m$. Also for $\alpha > 0$ we show that the only mass spectrum in $(m, 2m)$ is the bound state mass m_b.

Results analogous to ours for two-dimensional qft models in the Euclidean formulation have been obtained in refs. 7 and 8 and we use similar methods.

We now turn to a more precise description of the class of models we treat and of our results.

We let $s(x) \in R$, $x = (x_0, \vec{x}) \in \Lambda \subset Z^d$ denote the spin variable at the site x of the finite lattice Λ. For the generating function $Z_\Lambda(J)$ we take $Z_\Lambda(J) = \int e^{(J, s)} e^{S(s)}\, d\mu(s)$; $(J, s) = \sum_x J(x)\, s(x)$ and the interacting action $S(s)$ is $S(s) = \beta \sum' s(x)\, s(y)$ where $\sum'$ denotes the sum over the unordered set of nearest neighbor sites $\{x, y\}$. $d\mu(s) = \prod_x e^{-V(s(x))}\, ds(x)$ and we only consider the case of even ssd. i.e., $V(s) = V(-s)$. $V(s)$ is bounded from below and increases at infinity at least quadratically. Expectations of the probability measure $\exp[S(s)]\, d\mu(s)/$ normalization are denoted by $\langle \cdot \rangle_\Lambda$. Truncated cf's are given by local derivatives with respect to J's of $\ln Z_\Lambda(J)$ at $J = 0$.

By the polymer expansion (see ref. 4) the thermodynamic limit ($\Lambda \to Z^d$) of the cf's exist. The limiting cf's are denoted by $\langle \cdot \rangle$ and are translation invariant. The truncated cf's have exponential tree decay with a rate of at least $(1-\varepsilon)\, |\ln \beta|$ where $\varepsilon \to 0$ as $\beta \to 0$.

For a class of models defined by imposing conditions on V in the ssd Gaussian domination inequalities for correlation functions (cf) (see refs. 1, 9–11) have been proven which exclude bound state spectrum for $\beta \geqslant 0$. Taking coincident points and setting $\beta = 0$ these cf inequalities become inequalities on the moments of the ssd given by $\langle s^{2n} \rangle \leqslant (2n-1)(2n-3) \cdots (3)(1) \langle s^2 \rangle^n$. For small β our result, with the exception of the Ising model, extends to a wider class of models than those of refs. 4, 9–11 as the conditions we impose are $\langle s^4 \rangle - 3\langle s^2 \rangle^2 < 0$ and $\langle s^2 \rangle^2 < \langle s^4 \rangle$. Note that $\langle s^2 \rangle^2 \leqslant \langle s^4 \rangle$ by the Cauchy–Schwarz inequality.

Associated with the model is an imaginary discrete time lattice quantum field theory (qft). The qft is constructed in the standard way (see refs. 1 and 2). Taking the x_0 direction as time the construction provides the quantum mechanical Hilbert space H with inner product $(\cdot, \cdot)$, commuting self-adjoint energy-momentum (em) operators $H \geqslant 0$, $\vec{P}$, the time-zero field operator $\hat{s}(x)$, $x = (0, \vec{x})$ and the vacuum vector Ω. The relation of the

Hilbert space objects to the cf's is given by the Feynman–Kac (F–K) formula, i.e., setting $\hat{s}(0)=\hat{s}$, $x_k=(t_k, \vec{x}_k)$, with $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$,

$$(\Omega, \hat{s}e^{-H(t_2-t_1)}e^{i\vec{P}\cdot(\vec{x}_2-\vec{x}_1)}\hat{s}e^{-H(t_3-t_2)}e^{i\vec{P}\cdot(\vec{x}_3-\vec{x}_2)}\hat{s}\cdots e^{-H(t_n-t_{n-1})}e^{i\vec{P}\cdot(\vec{x}_n-\vec{x}_{n-1})}\hat{s}\Omega)$$
$$=\langle s(x_1)\cdots s(x_n)\rangle \tag{1.1}$$

We will state our main result in terms of the spectrum of H, $\vec{P}$ but first we give some known or easily obtained results on the e-m spectrum which we need here. We let $(E, \vec{p})$, $E \geqslant 0$, $\vec{p} \in T_{d-1}$ (the $d-1$-dimensional torus) denote the spectral parameters associated with $(H, \vec{P})$ and refer to the spectral points $(E, \vec{p}=\vec{0})$ as the mass spectrum.

The one-particle states are generated by vectors of the form $\hat{s}(\vec{x})\,\Omega$ and by the methods of ref. 4 have mass $m \sim \ln\beta$ for β small and an isolated real analytic dispersion curve $w(\vec{p}) \geqslant w(\vec{0}) \equiv m$. The e-m dispersion curve is determined as the zero of $\tilde{\Gamma}(p_0=iw(\vec{p}), \vec{p})$ where $\tilde{\Gamma}(p)$ is the Fourier transform of $\Gamma(x, y)$. Throughout this paper we define the Fourier transform without factors of 2π. $\Gamma(x, y)$ is minus the convolution inverse of the two-point function $\langle s(x)\, s(y)\rangle = S(x, y)$. To lowest order in β

$$w(\vec{p}) = -\ln\beta - \ln\langle s^2\rangle - 2\beta(d-1)\langle s^2\rangle + \beta\langle s^2\rangle\, 2\sum_{i=1}^{d-1}(1-\cos p_i) + O(\beta^2)$$

Furthermore there is no spectrum up to $-(2-\varepsilon)\ln\beta$, $\varepsilon(\beta)>0$, and $\varepsilon(\beta)\downarrow 0$ as $\beta\downarrow 0$. This is known as the upper mass gap property and implies the Orstein–Zernike behavior for the two-point function (see ref. 3).

To determine the mass spectrum (e-m spectrum at $\vec{p}=0$) in the interval $(m, 2m)$ we consider the states in the subspace generated by $\hat{s}(\vec{x})\,\hat{s}(\vec{y})\,\Omega$. The truncated 4-point function related to this state (after subtracting out the vacuum contribution) is

$$D(x_1 x_2; x_3 x_4) = \langle s(x_1)\, s(x_2)\, s(x_3)\, s(x_4)\rangle - \langle s(x_1)\, s(x_2)\rangle\langle s(x_3)\, s(x_4)\rangle$$

where $x_i=(t_i, \vec{x}_i)$. By translation invariance D depends only on the difference variables. We now introduce the newly-devised relative coordinates (ξ, η, τ) which are the substitute for the center of mass and relative coordinates used in the continuum. Let $\xi = x_2 - x_1$, $\eta = x_4 - x_3$, $\tau = x_3 - x_2$ and we denote by p, q, k the respective Fourier transform variables. Writing $\xi=(\xi_0, \vec{\xi})$, etc. it follows that if $\xi_0=\eta_0=0$ $D(\xi,\eta,\tau)=(\theta(-\vec{\xi}), e^{-H|\tau_0|}e^{i\vec{P}\vec{\tau}}\theta(\vec{\eta}))$ where $\theta(\vec{\eta})=\hat{s}(\vec{0})\,\hat{s}(\vec{\eta})\,\Omega - (\Omega, \hat{s}(\vec{0})\,\hat{s}(\vec{\eta})\,\Omega)\,\Omega$. A calculation shows, with $f\colon Z^{d-1}\to C$ and letting $\hat{D}(\vec{\xi}, \vec{\eta}, k)$ denote the Fourier transform in the τ variable only,

$$\iint \bar{f}(\vec{\xi})\, \hat{D}(\vec{\xi}, \vec{\eta}, k)\, f(\vec{\eta})\, d\vec{\xi}\, d\vec{\eta}$$

$$= (2\pi)^{d-1} \int_0^\infty \int_{T_{d-1}} \frac{\sinh E}{\cosh E - \cos k_0}\, d(\theta(f), \mathsf{E}(E, \vec{k})\, \theta(f)) \tag{1.2}$$

where $\mathsf{E}(E, \vec{q})$ is the spectral family associated with H, $\vec{P}$ and T^{d-1} is the $d-1$-dimensional torus, $\theta(f) = \sum_{\vec{x}} f(\vec{x})\, \theta(-\vec{x})$, $\vec{x} \in Z^{d-1}$. The singularities in k_0, for $\vec{k}$ fixed, of the left side are points in the e-m spectrum by considering the right side.

Our first main result we state as

Theorem 1. For $\beta > 0$ and sufficiently small and in the two-particle subspace

(a) for $\alpha < 0$ there is no mass spectrum in $(0, 2m)$,

(b) for $\alpha > 0$ the mass spectrum in $(0, 2m)$ consists of the single point $m_b = 2m + \ln(1 - \gamma) + O(\beta)$.

To prove this result we introduce a B–S equation which in operator form is

$$D = D_0 + DKD_0, \qquad K = D_0^{-1} - D^{-1}$$

or in terms of kernels is, with $x_{10} = x_{20}$, $x_{30} = x_{40}$,

$$D(x_1 x_2 x_3 x_4) = D_0(x_1 x_2 x_3 x_4) + \int D(x_1 x_2 y_1 y_2)\, \delta(y_{10} - y_{20})$$
$$\times K(y_1 y_2 y_3 y_4)\, \delta(y_{30} - y_{40})\, D_0(y_1 y_2 x_3 x_4)\, dy_1\, dy_2\, dy_3\, dy_4 \tag{1.3}$$

where

$$D_0(x_1 x_2 x_3 x_4) = \langle s(x_1)\, s(x_3)\rangle \langle s(x_2)\, s(x_4)\rangle + \langle s(x_1)\, s(x_4)\rangle \langle s(x_2)\, s(x_3)\rangle$$

and we use an integral notation for lattice sums and the Kronecker delta. K is called the B–S kernel. D, D_0 and K are considered as matrix operators acting in $s\ell_2(A)$, the symmetric subspace of $\ell_2(A)$, where $A = \{(x_1, x_2) \in Z^{2d} \mid x_{10} = x_{20}\}$. Crucial for the proof of Theorem I and for the proof of the bound state existence results of ref. 6 is a bound on K which is our second main result given by

Theorem II. For β sufficiently small, with $x_{10}=x_{20}$, $x_{30}=x_{40}$,

$$|K(x_1x_2x_3x_4)| \leqslant c_1 \left|\frac{\beta}{c_2}\right|^{3\,|x_{30}-x_{20}|+1/2[\,\|\vec{x}_1+\vec{x}_2-\vec{x}_3-\vec{x}_4\|+\|\vec{x}_2-\vec{x}_1\|+\|\vec{x}_4-\vec{x}_3\|\,]}$$

where $\|\vec{x}\| = \sum_{k=1}^{d-1} |x_k|$.

We give some intuition about the result and the method of proof in the context of lattice Schroedinger operators in $\ell_2(Z^{d-1})$. Consider the B–S equation in relative coordinates. The Fourier transforms in the time variable with spectral parameter $k_0=i\chi$, $\chi=2m+z$, $\operatorname{Re} z<0$, is, roughly speaking, a two-body Schroedinger resolvent equation (in $\ell_2(Z^{d-1})$)

$$(H-z)^{-1}=(H_0-z)^{-1}-(H-z)^{-1}\,V(H_0-z)^{-1}$$

where $H_0 \approx \beta\Delta$ with $-\Delta$ the lattice Laplacian. $V=\lambda\delta+W$ where δ is the delta function potential and $\lambda>0$ so we have a repulsive potential. W is not local but has exponentially decaying kernel and is of order β^2. Now the kernel of $(H'-z)^{-1}$ where $H'=H_0+\lambda\delta$ can be obtained explicitly and doesn't blow up as $\operatorname{Re} z \uparrow 0$. Taking $(H'-z)^{-1}$ as the unperturbed resolvent one shows that $(f, (H-z)^{-1} f)$ has a convergent Neumann series uniformly in $\operatorname{Re} z<0$ for f in a dense set. In the attractive case ($\lambda<0$) there is an isolated point in the spectrum corresponding to a multiplicity one eigenvalue at $z=z_b<0$ with isolation radius b but the Neumann series still converges for $z_b+b<z<0$.

We now describe the organization of this paper. In Section II we prove Theorem II and in Section III the proof of Theorem I is given. Brief proofs will be given as the arguments that we use have previously appeared in many places (see ref. 4).

II. DECAY OF THE BETHE–SALPETER KERNEL

Here we use a decoupling of hyperplane method (see refs. 4 and 6) and prove the decay bound on K given in Theorem II. It is to be understood that we carry out the analysis in a finite volume. We will obtain bounds on cf's which are independent of the volume. For the bonds in the Boltzman factor between the hyperplanes $x^k=q$ and $x^k=q+1$, $k=0, 1,..., d-1$ replace β by the complex parameter w^k_q. For β_0 small by the polymer expansion the cf are jointly analytic in β and the w^k_q's for $|\beta|, |w^k_q| \leqslant \beta_0$. By setting $w^k_q=\beta \leqslant \beta_0$ for all q and k we recover the physical translationally invariant cf. For a matrix operator $M(x, y)$ in ℓ_2 the ℓ_2 operator norm is denoted by $|M|$ which is bounded by $\{\sup_y[\sum_x |M(x, y)|]\}^{1/2} \cdot \{\sup_x [\sum_y |M(x, y)|]\}^{1/2}$. Recall that we are considering D and D_0 as matrix operators in $s\ell_2(A)$. By the exponential decay of the two-point function we see that D_0 has finite norm. D can be written as $D = S_4^T + D_0$ where S_4^T is the truncated four-point function which has exponential tree decay. Thus D also has finite norm. We now consider the inverses of D_0 and D. If M is D_0 or D we decompose M as $M = M_d + M_n$ where M_d is diagonal and M_n is non-diagonal. The inverse M^{-1} is defined by the Neumann series $M^{-1} = M_d^{-1} \sum_{k=0}^{\infty} (-1)^k [M_n M_d^{-1}]^k$. Expanding D_{0d} and D_d in β shows that D_{0d} and D_d are invertible with bounds

$$|D_{0d}^{-1}| \leqslant \langle s^2 \rangle^{-2}, \qquad |D_d^{-1}| \leqslant \max\{2(\langle s^4 \rangle - \langle s^2 \rangle^2)^{-1}, \langle s^2 \rangle^{-2}\}$$

As $|D_{0n}| \leqslant c\,|\beta|$ and $|D_n| \leqslant c\,|\beta|^{1/2}$ we see that D^{-1}, D_0^{-1} and thus also $K = D_0^{-1} - D^{-1}$ exist as bounded operators on $s\ell_2$.

First consider the decay of K in the time direction. We will need w_q^0 derivatives of D and D_0 which we give in the lemma below. We set $e^0 = e$ and for notational simplicity drop the superscript on w_q^0 and write $\partial \equiv \partial/\partial w_q$. We have

Lemma II.1. For $x_1^0 = x_2^0 \leqslant q < x_3^0 = x_4^0$,

$$D(x_1 x_2 x_3 x_4)|_{w_q=0} = 0, \qquad \partial D(x_1 x_2 x_3 x_4)|_{w_q=0} = 0$$

$$\partial^2 D(x_1 x_2 x_3 x_4)|_{w_q=0} = \left\{ \sum_{z_1^0 = z_2^0 = q} D(x_1 x_2 z_1 z_2)\, D(z_1 + e z_2 + e x_3 x_4) \right\}\Bigg|_{w_q=0}$$

and the same for $D_0(x_1 x_2 x_3 x_4)$.

For the proof of the lemma and for other derivative calculations we refer the reader to refs. 4 and 6 where similar calculations are carried out. Let $H_1^p \subset s\ell_2$ be generated by $\{e_{(x,\,y)} : x^0 = y^0 \leqslant p\}$ and H_2^p be generated by $\{e_{(x,\,y)} : x^0 = y^0 > p\}$ so that $s\ell_2 = H_1^p \oplus H_2^p$. By Lemma II.1 D_0 and D at $w_q = 0$ leave H_i^q, $i = 1, 2$ invariant and the same holds for D_0^{-1} and D^{-1} at $w_q = 0$. Concerning the w_q derivatives of K we have

Lemma II.2. For $x_1^0 = x_2^0 \leqslant q < x_3^0 = x_4^0$

(a) $\partial K(x_1 x_2 x_3 x_4)|_{w_q=0} = 0$,

(b) $\partial^2 K(x_1 x_2 x_3 x_4)|_{w_q=0} = 0$.

Proof. (a) $\partial K = -D_0^{-1}\, \partial D_0\, D_0^{-1} + D^{-1}\, \partial D\, D^{-1}$ and using Lemma II.1 the result follows.

(b) $\partial^2 K = 2 D_0^{-1}\, \partial D_0\, D_0^{-1}\, \partial D_0\, D_0^{-1} - D_0^{-1}\, \partial^2 D_0\, D_0^{-1} - 2 D^{-1}\, \partial D\, D^{-1}$ $\partial D.\ D^{-1} + D^{-1}\, \partial^2 D\, D^{-1} \equiv k_1 + k_2 + k_3 + k_4$.

At $w_q = 0$ k_1 and k_3 leave H_i^q, $i = 1, 2$ invariant and therefore are zero. Using Lemma II.1 for $\partial^2 D_0$ and $\partial^2 D$ at $w_q = 0$ in k_2 and k_4 shows that $k_2 + k_4 = 0$.

Using Lemma II.2, joint analyticity in $\{w_q\}$ and Cauchy bounds for w_q derivatives we have the bound $|K(x_1 x_2 x_3 x_4)| \leqslant c_1\, |\beta/c_2|^{3\, |x_3^0 - x_2^0|}$, $|x_3^0 - x_2^0| \geqslant 1$.

We now turn to the bound in the spatial directions. We can consider each direction independently and the same analysis applies to each direction. Thus, without loss of generality, let us consider the 1-direction. There are 24 regions to be considered, one for each order of x_1', x_2', x_3', x_4', where the superscript refers to the 1st component. By the $x_1 \leftrightarrow x_2$, $x_3 \leftrightarrow x_4$, $x_1, x_2 \leftrightarrow x_3, x_4$ invariance of $K(x_1 x_2 x_3 x_4)$ (which follows from the invariance of D_0 and D) the 24 regions partition into 3 groups of 8 regions. For each of the 3 groups the bound we obtain will also be invariant under the above substitutions.

Let

$$L_1^p = \text{span of } \{e_{(x,\, y)} : x',\, y' \leqslant p\}$$

$$L_{21}^p = \text{span of } \{e_{(x,\, y)} : x' \leqslant p < y' \text{ or } y' \leqslant p < x'\}$$

$$L_{31}^p = \text{span of } \{e_{(x,\, y)} : x',\, y' > p\}$$

Consider first $x_1' \leqslant x_2' \leqslant x_3' \leqslant x_4'$. If $x_1' \leqslant p < x_2' \leqslant x_3' \leqslant x_4'$ or $x_1' \leqslant x_2' \leqslant p < x_3' \leqslant x_4'$ then for $w_p = 0$ $D_0 = D = 0$. Thus $w_p = 0$ D and D_0 leave L_3^p invariant. Likewise if $x_1' \leqslant x_2' \leqslant x_3' \leqslant p < x_4'$ $D_0 = D = 0$ at $w_p = 0$. This implies $(f, D_0|_{w_p = 0}\, g) = 0$ for all $f \in L_1^p$ and $g \in L_2^p$. If $f \in L_3^p$ then since $D_0|_{w_p = 0}$ is symmetric $(f, D_0|_{w_p=0}\, g) = (\bar{D}_0|_{w_p=0}\, f, g) = 0$. Thus $D_0|_{w_p=0}$ leaves L_1^p invariant. The same conclusion holds for $D|_{w_p=0}$ and therefore also for $D_0^{-1}|_{w_p=0}$ and $D^{-1}|_{w_p=0}$. Thus $K|_{w_p=0}$ leaves each L_i^p invariant which for $x_1' \leqslant x_2' \leqslant x_3' \leqslant x_4'$ implies the bound $|K(x_1 x_2 x_3 x_4)| \leqslant c_1\, |\beta/c_2|^{(x_2' - x_1') + (x_3' - x_2') + (x_4' - x_3')}$. Since

$$\begin{aligned} &\tfrac{1}{2}\, |x_1' + x_2' - x_3' - x_4'| + \tfrac{1}{2}\, |x_2' - x_1'| + \tfrac{1}{2}\, |x_4' - x_3'| \\ &\qquad \leqslant |x_2' - x_1'| + |x_3' - x_2'| + |x_4' - x_3'| \end{aligned}$$

we have

$$|K(x_1 x_2 x_3 x_4)| \leqslant c_1 \left| \frac{\beta}{c_2} \right|^{1/2[\, |x_1' + x_2' - x_3' - x_4'| + |x_2' - x_1'| + |x_4' - x_3'|\,]} \tag{2.1}$$

which also holds in the symmetry related regions $x_2' \leqslant x_1' \leqslant x_3' \leqslant x_4'$; $x_1' \leqslant x_2' \leqslant x_4' \leqslant x_3'$; $x_2' \leqslant x_1' \leqslant x_4' \leqslant x_3'$; $x_3' \leqslant x_4' \leqslant x_1' \leqslant x_2'$; $x_4' \leqslant x_3' \leqslant x_1' \leqslant x_2'$;

$x'_3 \leqslant x'_4 \leqslant x'_2 \leqslant x'_1$ and $x'_4 \leqslant x'_3 \leqslant x'_2 \leqslant x'_1$. For the region $x'_1 \leqslant x'_3 \leqslant x'_2 \leqslant x'_4$ if $x'_1 \leqslant p < x'_3 \leqslant x'_2 \leqslant x'_4$ then $K(x_1 x_2 x_3 x_4)|_{w_q=0} = 0$ as $e_{(x_1 x_2)} \in L^p_2$ and $e_{(x_3 x_4)} \in L^p_3$. If $x'_1 \leqslant x'_3 \leqslant p < x'_2 \leqslant x'_4$ then

$$D_0|_{w_p=0} = \langle s(x_1)\, s(x_3)\rangle \langle s(x_2)\, s(x_4)\rangle|_{w_p=0} = D|_{w_p=0}$$

which means for the restrictions to L^p_2 $D_0|_{w_p=0} = D|_{w_p=0}$ and thus $K|_{w_p=0} = 0$. If $x'_1 \leqslant x'_3 \leqslant x'_2 \leqslant p < x'_4$ then $K|_{w_p=0} = 0$ as $e_{(x_1 x_2)} \in L^p_1$ and $e_{(x_3 x_4)} \in L^p_2$. These considerations lead to the bound $|K(x_1 x_2 x_3 x_4)| \leqslant c_1\, |\beta/c_2|^{(x'_3 - x'_1) + (x'_2 - x'_3) + (x'_4 - x'_2)}$ and since $\frac{1}{2}|x'_1 + x'_2 - x'_3 - x'_4| + \frac{1}{2}|x'_2 - x'_1| + \frac{1}{2}|x'_4 - x'_3| \leqslant |x'_3 - x'_1| + |x'_2 - x'_3| + |x'_4 - x'_2|$ we again get the bound (2.1) which extends to the other 7 symmetry related regions. Finally the region $x'_1 \leqslant x'_3 \leqslant x'_4 \leqslant x'_2$ is treated in a similar way to the first group of regions with the resulting bound $|K(x_1 x_2 x_3 x_4| \leqslant c_1\, |\beta/c_2|^{(x'_3 - x'_1) + (x'_4 - x'_3) + (x'_2 - x'_4)}$ and as

$$\frac{1}{2}|x'_1 + x'_2 - x'_3 - x'_4| + \frac{1}{2}|x'_2 - x'_1| + \frac{1}{2}|x'_4 - x'_3|$$
$$\leqslant |x'_3 - x'_1| + |x'_4 - x'_3| + |x'_2 - x'_1|$$

the bound (2.1) holds and extends to the other 7 symmetry related regions. Repeating the analysis for each of the other spatial directions and putting it together with the temporal bound gives us the bound of the theorem.

III. BOUND STATES

Here we prove Theorem I. Concerning the proof of (b) for $\alpha > 0$ it has been shown in ref. 6 that there is a bound state with mass m_b and that this is the only point in the spectrum in $(0, m_b + \delta\gamma)$ in the two-particle subspace for δ sufficiently small. By an easy modification of the proof of (a) we give a simple short proof of the absence of mass spectrum in $[m_b + \delta\gamma, 2m)$. We now turn to the proof of (a). Referring to Eq. (1.2) we show the absence of mass spectrum in $(0, 2m)$ by showing there is no singularity in $\chi \in (0, 2m)$, $k = (k_0 = i\chi, \vec{0}) \equiv k^0$ of

$$(f, \hat{D}f)_2 \equiv \int \bar{f}(\vec{\xi})\, \hat{D}(\vec{\xi}, \vec{\eta}, k^0)\, f(\vec{\eta})\, d\vec{\xi}\, d\vec{\eta} \tag{3.1}$$

In Eq. (3.1) ^ denotes the Fourier transform of $D(\vec{\xi}, \vec{\eta}, \tau)$ in the τ variable only and $f(\vec{\xi})$ is taken to be in the even subspace of the weighted ℓ_2 space

$$\ell_\delta = \left\{ f\colon Z^{d-1} \to C \;\middle|\; \int |f(\vec{\xi})|^2\, e^{\delta |\vec{\xi}|}\, d\vec{\xi} < \infty,\ \delta = \frac{m}{8} \right\}$$

For functions and operators in this space we use a subscript δ for their norms.

We introduce a B–S equation by $D = D_0 + DKD_0$ or in terms of kernels, with $x_{10} = x_{20}$, $x_{30} = x_{40}$,

$$D(x_1 x_2 x_3 x_4) = D_0(x_1 x_2 x_3 x_4) + \int D(x_1 x_2 y_1 y_2)\, \delta(y_{10} - y_{20})$$
$$\times K(y_1 y_2 y_3 y_4)\, \delta(y_{30} - y_{40})\, D_0(y_3 y_4 x_3 x_4)\, dy_1\, dy_2\, dy_3\, dy_4 \tag{3.2}$$

where we use a continuum notation for lattice sums and the Kronecker delta. Using the invariance of D, D_0 and consequently also K under the substitutions $(x_1 x_2 x_3 x_4) \to (x_2 x_1 x_3 x_4)$ and $(x_1 x_2 x_3 x_4) \to (x_1 x_2 x_4 x_3)$ we can write the B–S equation in relative coordinates as, without changing notation for functions written in relative coordinates,

$$D(\vec{\xi}, \vec{\eta}, \tau) = D_0(\vec{\xi}, \vec{\eta}, \tau) + \int d\vec{\xi}'\, d\vec{\eta}'\, d\tau'\, d\tau''\, D(\vec{\xi}, \vec{\xi}', \tau')$$
$$\times K(-\vec{\xi}', -\vec{\eta}', \tau - \tau' - \tau'')\, D_0(\vec{\eta}', \vec{\eta}, \tau) \tag{3.3}$$

Upon taking the Fourier transform of Eq. (3.3) in the τ variable only we obtain

$$\hat{D}(\vec{\xi}, \vec{\eta}, k) = \hat{D}_0(\vec{\xi}, \vec{\eta}, k) + \int d\vec{\xi}'\, d\vec{\eta}'\, \hat{D}(\vec{\xi}, \vec{\xi}', k)\, \hat{K}(-\vec{\xi}', -\vec{\eta}', k)\, \hat{D}_0(\vec{\eta}', \vec{\eta}, k) \tag{3.4}$$

and for $k = k^0$, $\hat{K}(-\vec{\xi}, -\vec{\eta}, k^0) = \hat{K}(\vec{\xi}, \vec{\eta}, k^0)$.

In terms of relative coordinates Theorem II furnishes us with the bound

$$|K(\vec{\xi}, \vec{\eta}, \tau)| \leqslant c_1\, |\beta/c_2|^{3\,|\tau_0| + 1/2[\,|2\vec{\tau} + \vec{\xi} + \vec{\eta}| + |\vec{\xi}| + |\vec{\eta}|\,]} \tag{3.5}$$

For $k = k^0$ we write Eq. (3.4) in operator form as

$$\hat{D}(k^0) = \hat{D}_0(k^0) + \hat{D}(k^0)\, \hat{K}(k^0)\, \hat{D}_0(k^0) \tag{3.6}$$

to control $\hat{D}(k^0)$ in Eq. (3.6) we first decompose K as $K = L + M$ where

$$\begin{aligned} L(\vec{\xi}, \vec{\eta}, \tau) &= \rho\delta(\vec{\xi})\, \delta(\vec{\eta})\, \delta(\tau) \\ \rho &= (2\langle s^2\rangle^2)^{-1}\, (\langle s^4\rangle - \langle s^2\rangle^2)^{-1}\, (\langle s^4\rangle - 3\langle s^2\rangle^2) \end{aligned} \tag{3.7}$$

and we assume $\langle s^4 \rangle > \langle s^2 \rangle^2$ and $\rho < 0$. L, which is local and β independent, is called the ladder approximation and is obtained by expanding K in powers of β and keeping only the constant term. Next we define

$$\hat{D}'_0 = \hat{D}_0 (I - \hat{L}\hat{D}_0)^{-1} \tag{3.8}$$

We can explicitly solve for the kernel of $\hat{D}'_0$ obtaining

$$\hat{D}'_0(\vec{\xi}, \vec{\eta}, k_0) = \hat{D}_0(\vec{\xi}, \vec{\eta}, k_0) + \rho(1 - \rho\hat{D}_0(\vec{0}, \vec{0}, k^0))^{-1} \hat{D}_0(\vec{\xi}, \vec{0}, k^0)\, \hat{D}_0(\vec{0}, \vec{\eta}, k^0) \tag{3.9}$$

and in terms of $\hat{D}'_0$

$$\hat{D} = \hat{D}'_0 (I - \hat{M}\hat{D}'_0)^{-1} \tag{3.10}$$

We treat $\hat{D}$ as well as $\hat{D}'_0$ as operators from $\ell_\delta \to \ell_{-\delta}$ and $\hat{K}: \ell_{-\delta} \to \ell_\delta$ so that Eq. (3.1) can be written as $\langle f, \hat{D}f \rangle$, the evaluation of the linear functional $\hat{D}f$ acting on f, where $f \in \ell_\delta$ and $\hat{D}f$ is in the dual space $\ell_{-\delta}$. For $\hat{D}'_0: \ell_\delta \to \ell_{-\delta}$ the norm is equal to the ℓ_2 norm of $(\hat{D}'_0)_\delta$ where $(\hat{D}'_0)_\delta$ has the kernel $e^{-\delta |\vec{\xi}|} (\hat{D}'_0)(\vec{\xi}, \vec{\eta}, k^0)\, e^{-\delta |\vec{\eta}|}$.

For sufficiently small β and for $\chi \in (0, 2m)$ we show that $\hat{D}$ exists by showing that the $\ell_\delta \to \ell_{-\delta}$ norm of $\hat{D}'_0$ is bounded and that the ℓ_δ norm of $\hat{M}\hat{D}'_0$ is less than one which in turn depends on bounds on $\hat{D}'_0$ and improved bounds on M for some special small distance points.

In order to bound $\hat{D}'_0$ of Eq. (3.9) we use a representation for D_0 obtained by using the spectral representation of the two-point function given by, for small β,

$$S(x) = \int_0^\infty \int_{T_{d-1}} e^{-E |x_0|} e^{i\vec{p} \cdot \vec{x}}\, d\sigma_{\vec{p}}(E)\, d\vec{p} \tag{3.11}$$

where

$$d\sigma_{\vec{p}}(E) = Z(\vec{p}, \beta)\, \delta(E - w(\vec{p}))\, dE + d\hat{\sigma}_{\vec{p}}(E)$$
$$w(\vec{p}) = -\ln \beta - \ln \langle s^2 \rangle + r(\beta, \vec{p})$$

$d\sigma_{\vec{p}}(E) = d\sigma_{-\vec{p}}(E)$ as well as $d\hat{\sigma}_{\vec{p}}(E) = d\hat{\sigma}_{-\vec{p}}(E)$ are positive measures and $Z(\vec{p}, \beta) = \langle s^2 \rangle / (2\pi)^{d-1} + O(\beta)$. $d\sigma_{\vec{p}}(E)$ has support in $((3 - \varepsilon')\, m, \infty)$ where $\varepsilon' \downarrow 0$ as $\beta \downarrow 0$; $r(\vec{p}, \beta) = O(\beta)$ is jointly analytic in β and $\vec{p}$. These results are obtained by adapting the work of refs. 2 and 4. Furthermore we have

Lemma 3.1. $\hat{\sigma}_{\vec{p}}(R^+) \equiv \int_0^\infty d\hat{\sigma}_{\vec{p}}(E) = O(\beta)$.

Proof. Using the spectral representation for $S(x_0=0, \vec{x})$ and taking the spatial Fourier transform gives

$$S(x=0) + \sum_{|\vec{x}| \geqslant 1} e^{-i\vec{q}\cdot\vec{x}} S(0, \vec{x}) = (2\pi)^{d-1} \int_0^\infty d\sigma_{\vec{q}}(E)$$

$$= (2\pi)^{d-1} Z(\vec{q}, \beta) + (2\pi)^{d-1} \int_0^\infty d\hat{\sigma}_{\vec{q}}(E)$$

As $S(x=0) = \langle s^2 \rangle + O(\beta^2)$, $|S(0, \vec{x})| \leqslant c_1 |\beta/c_2|^{|\vec{x}|}$ and $Z(\vec{q}, \beta) = \langle s^2 \rangle / (2\pi)^{d-1} + O(\beta)$ the result follows.

Using Eq. (3.11) we obtain a representation for $\hat{D}_0(\vec{\xi}, \vec{\eta}, k^0) = \sum_\tau D_0(\vec{\xi}, \vec{\eta}, \tau)\, e^{-ik_0\tau_0}$, where $\hat{D}_0(\vec{\xi}, \vec{\eta}, \tau) = S(\tau + \vec{\xi})\, S(\tau + \vec{\eta}) + S(\tau)\, S(\tau + \vec{\xi} + \vec{\eta})$, given by

$$\hat{D}_0(\vec{\xi}, \vec{\eta}, k^0) = 2(2\pi)^{d-1} \int_0^\infty \int_0^\infty \int_{T_{d-1}} \frac{\sinh(E+E')}{\cosh(E+E') - \cosh\chi} \times \cos \vec{p}\cdot\vec{\eta} \cos \vec{p}\cdot\vec{\xi}\, d\sigma_{\vec{p}}(E)\, d\sigma_{\vec{p}}(E')\, d\vec{p} \tag{3.12}$$

Concerning the bounds on $\hat{D}_0'$ we write

$$\hat{D}_0'(\vec{\xi}, \vec{\eta}, k^0) = \hat{D}_{0a}'(\vec{\xi}, \vec{\eta}, k^0) + \hat{D}_{0b}'(\vec{\xi}, \vec{\eta}, k^0)$$

where

$$\hat{D}_{0a}'(\vec{\xi}, \vec{\eta}, k^0) = (1 - \rho \hat{D}_0(\vec{0}, \vec{0}, k^0))^{-1}\, \hat{D}_0(\vec{\xi}, \vec{\eta}, k^0)$$

$$\hat{D}_{0b}'(\vec{\xi}, \vec{\eta}, k^0) = -\rho(1 - \rho \hat{D}_0(\vec{0}, \vec{0}, k^0))^{-1} [\hat{D}_0(\vec{\xi}, \vec{\eta}, k^0)\, \hat{D}_0(\vec{0}, \vec{0}, k^0) - \hat{D}_0(\vec{\xi}, \vec{0}, k^0)\, \hat{D}_0(\vec{0}, \vec{\eta}, k^0)]$$

and we have

Lemma 3.2. For small β and uniformly for $\chi \in (0, 2m)$

(a) $|\hat{D}_{0a}'(\vec{\xi}, \vec{\eta}, k^0)| \leqslant |\rho|^{-1}$,

(b) $|\hat{D}_{0b}'(\vec{\xi}, \vec{\eta}, k^0)| \leqslant O(1)[1 - \delta(\vec{\xi})][1 - \delta(\vec{\eta})][1 + \beta^{-1} |\vec{\eta}|^2]$,

(c) $|(\hat{D}_0')_\delta|_2 \leqslant c\beta^{-1}$.

Proof. (a) From Eq. (3.12) $|\hat{D}_0(\vec{\xi}, \vec{\eta}, k^0)| \leqslant \hat{D}_0(\vec{0}, \vec{0}, k^0)$ and as $\rho < 0$ the results follows. (b) Letting $\Phi(\vec{\xi}, \vec{\eta}, k^0) \equiv \hat{D}_0(\vec{\xi}, \vec{\eta}, k^0)\, \hat{D}_0(\vec{0}, \vec{0}, k^0) - \hat{D}_0(\vec{\xi}, \vec{0}, k^0)\, \hat{D}_0(\vec{0}, \vec{\eta}, k^0)$ and using Eq. (3.12) we have

$$|\Phi(\vec{\xi}, \vec{\eta}, k^0)| \leqslant 4(2\pi)^{2(d-1)} \int \cdots \int \left[\frac{\sinh(E+E')}{\cosh(E+E') - \cosh \chi} \right]$$

$$\times \left[\frac{\sinh(F+F')}{\cosh(F+F') - \cosh \chi} \right] |\cos \vec{p} \cdot \vec{\eta} - \cos \vec{p}\,' \cdot \vec{\eta}|$$

$$\times d\sigma_{\vec{p}}(E)\, d\sigma_{\vec{p}}(E')\, d\sigma_{\vec{p}\,'}(F)\, d\sigma_{\vec{p}\,'}(F')\, d\vec{p}\, d\vec{p}\,'$$

Using $|\cos \vec{p} \cdot \vec{\eta} - \cos \vec{p}\,' \cdot \vec{\eta}| \leqslant (1 - \cos \vec{p} \cdot \vec{\eta}) + (1 - \cos \vec{p}\,' \cdot \vec{\eta})$ we get

$$|\Phi(\vec{\xi}, \vec{\eta}, k^0)| \leqslant \hat{D}_0(\vec{0}, \vec{0}, k^0)\, 4(2\pi)^{d-1} \iint \left[\frac{\sinh(E+E')}{\cosh(E+E') - \cosh \chi} \right]$$

$$\times (1 - \cos \vec{p} \cdot \vec{\eta})\, d\sigma_{\vec{p}}(E)\, d\sigma_{\vec{p}}(E')\, d\sigma_{\vec{p}}\, d\vec{p}$$

Using the decomposition of $d\sigma_{\vec{p}}(\cdot)$ and noting that the contributions involving $d\hat{\sigma}_{\vec{p}}(\cdot)$ are $O(1)$ we have

$$|\hat{D}'_{0b}(\vec{\xi}, \vec{\eta}, k^0)|$$

$$\leqslant 4(2\pi)^{d-1} \left[O(1) + \int \frac{\sinh 2w(\vec{p})}{\cosh 2w(\vec{p}) - \cosh \chi} (1 - \cos \vec{p} \cdot \vec{\eta})\, Z(\vec{p})^2\, d\vec{p} \right]$$

Taking into account that $Z(\vec{p}) = \langle s^2 \rangle / (2\pi)^{d-1} + O(\beta)$ the last integral is bounded by $c\beta^{-1} |\vec{\eta}|^2$. (c) Follows from (a) and (b) using $|(\hat{D}'_0)_\delta|_2 \leqslant \sup_{\vec{\xi}} [e^{-\delta |\vec{\xi}|} \sum_{\vec{\eta}} |\hat{D}'_0(\vec{\xi}, \vec{\eta}, k^0)|\, e^{-\delta |\vec{\eta}|}]$.

With regard to the bounds on M we have, denoting by e_1, the unit vector in the 1-direction,

Lemma 3.3. For small β and uniformly for $\chi \in (0, 2m)$

(a) $\hat{M}(\vec{0}, \vec{0}, k^0) = O(\beta^{1/2})$,

(b) $K(e_1, -e_1, \vec{0}) = K(e_1, e_1, -e_1) = O(\beta^2)$.

Proof. (a) From Eq. (3.5) $|K(\vec{0}, \vec{0}, \tau)| \leqslant c_1 |\beta / c_2|^{3|\tau_0| + |\vec{\tau}|}$ and as $M(\vec{0}, \vec{0}, 0) = O(\beta)$ the result follows. (b) In terms of the x coordinates, taking $\vec{x}_1 = \vec{0}$, $x_2 = e_1$, $\vec{x}_3 = e_1$, $\vec{x}_4 = \vec{0}$, we have to bound $K(\vec{0}, e_1, e_1, \vec{0}) = K(\vec{0}, e_1, \vec{0}, e_1)$. Using $DD^{-1} = I$ and $D(\vec{0}, e_1, \vec{0}, e_1) = \langle s^2 \rangle^2 + O(\beta^2)$ shows that $D^{-1}(\vec{0}, e_1, \vec{0}, e_1) = \frac{1}{2} \langle s^2 \rangle^{-2} (1 + O(\beta^2))$. Similarly $D_0^{-1}(\vec{0}, e_1, \vec{0}, e_1) = \frac{1}{2} \langle s^2 \rangle^{-2} (1 + O(\beta^2))$ and the result follows. We can realize $(\vec{\xi}, \vec{\eta}, \vec{\tau}) = (e_1, e_1, -e_1)$ as $\vec{x}_1 = \bar{0}$, $x_2 = e_1$, $\vec{x}_3 = \vec{0}$, $\vec{x}_4 = e_1$ so that the first equality in (b) holds.

Towards showing $|\hat{M}\hat{D}'_0|_\delta < 1$ we note that if a matrix operator on ℓ_δ has the kernel $B(\vec{\xi}, \vec{\eta})$ then the Hilbert–Schmidt norm (denoted by $|B|_{\delta HS}$) is

$$\left(\int d\vec{\xi}\, d\vec{\eta}\, e^{2\delta\,|\vec{\xi}|}\,|B(\vec{\xi}, \vec{\eta})|^2\, e^{-2\delta\,|\vec{\eta}|}\right)^{1/2}$$

We have

Lemma 3.4. $|\hat{M}\hat{D}'_{0a}|_\delta \leqslant O(\beta^{1/4})$ uniformly for $\chi \in (0, 2m)$.

Proof. Consider $f(\vec{\xi}, k^0) \equiv \sum_{\vec{\eta}} |\hat{M}(\vec{\xi}, \vec{\eta}, k^0)|$. From Eq. (3.5) and Lemma 3.3a if $\vec{\xi} = \vec{0}$, $f(\vec{\xi}, k^0) = O(\beta^{1/2})$ and if $\vec{\xi} \neq \vec{0}$ $f(\vec{\xi}, k^0) = O(\beta^{1/4})$ $e^{-1/4(m-O(1))\,|\vec{\xi}|}$. Thus the bound $f(\vec{\xi}, k^0) = O(\beta^{1/2})\, e^{-1/4(m-O(1))\,|\vec{\xi}|}$ holds for all $\vec{\xi} \in Z^{d-1}$ uniformly for $\chi \in (0, 2m)$. Since $|\hat{D}'_{0a}(\vec{\xi}, \vec{\eta}, k^0)| \leqslant |\rho|^{-1}$ we have

$$|\hat{M}\hat{D}'_{0a}(\vec{\xi}, \vec{\eta})| \leqslant O(1) \sum_{\vec{\lambda}} |\hat{M}(\vec{\xi}, \vec{\lambda}, k^0)| \leqslant O(\beta^{1/8})\, e^{-1/4(m-O(1))\,|\vec{\xi}|}$$

which shows that $|\hat{M}\hat{D}'_{0a}|_{\delta HS} \leqslant O(\beta^{1/4})$ if $O(1) < \delta < m/8$.

Finally we have

Lemma 3.5. $|\hat{M}\hat{D}'_{0b}|_\delta \leqslant O(\beta^{1/8})$ uniformly for $\chi \in (0, 2m)$.

Proof. Using Lemma 3.2b we have

$$|\hat{M}(k^0)\, \hat{D}'_{0b}(\vec{\xi}, \vec{\eta})| \leqslant O(1)\, \beta^{-1} \sum_{\vec{\lambda} \neq 0} |\hat{K}(\vec{\xi}, \vec{\lambda}, k^0)|\, (1 - \delta(\vec{\eta}))\, |\vec{\eta}|^2$$

From Eq. (3.5) $|\hat{K}(\vec{\xi}, \vec{\lambda}, k^0)| \leqslant \sum_{\vec{\tau}} |K(\vec{\xi}, \vec{\lambda}, \vec{\tau})| + O(\beta)\, e^{-1/2(m-O(1))[\,|\vec{\xi}| + |\vec{\lambda}|\,]}$ if $\chi \in (0, 2m)$. Thus $R \equiv \sum_{\vec{\lambda} \neq \vec{0}} |\hat{K}(\vec{\xi}, \vec{\lambda}, k^0)| \leqslant \sum_{\vec{\lambda} \neq \vec{0}} \sum_{\vec{\tau}} |K(\vec{\xi}, \vec{\lambda}, \vec{\tau})| + O(\beta^{3/2})$ $e^{-1/2(m-O(1))\,|\vec{\xi}|}$. Now $\sum_{|\vec{\lambda}| \geqslant 2} \sum_{\vec{\tau}} |K(\vec{\xi}, \vec{\lambda}, \vec{\tau})| \leqslant O(\beta)\, e^{-1/2(m-O(1))\,|\vec{\xi}|}$ so that $R \leqslant \sum_{|\vec{\lambda}|=1} \sum_{\vec{\tau}} |K(\vec{\xi}, \vec{\lambda}, \vec{\tau})| + O(\beta)\, e^{-1/2(m-O(1))\,|\vec{\xi}|}$. Since $\min_{\vec{\tau} \in Z^{d-1}} |2\vec{\tau} + \vec{\lambda}| = 1$ if $|\vec{\lambda}| = 1$ we have $\sum_{|\vec{\lambda}|=1} \sum_{\vec{\tau}} |K(\vec{0}, \vec{\lambda}, \vec{\tau})| \leqslant O(\beta)$. Similarly if $|\vec{\lambda}| = 1$ and if $|\vec{\xi}| = 1$ but $\vec{\xi} \neq \pm\vec{\lambda}$, $\sum_{\vec{\tau}} |K(\vec{\xi}, \vec{\lambda}, \vec{\tau})| \leqslant O(\beta)\, e^{1/2(m-O(1))\,|\vec{\xi}|}$. Now suppose that $\vec{\xi} = -\vec{\lambda}$, $|\vec{\xi}| = 1$. We can take $\vec{\xi} = e_1 = -\vec{\lambda}$. Then $\sum_{\vec{\tau}} |K(e_1, -e_1, \vec{\tau})| = |K(e_1, -e_1, \vec{0})| + O(\beta^{3/2})\, e^{-1/2(m-O(1))\,|\vec{\xi}|}$ and using Lemma 3.2b $|K(e_1, -e_1, \vec{0})| = O(\beta^{3/2})\, e^{-1/2(m-O(1))\,|e_1|}$. Thus

$$\sum_{\vec{\tau}} |K(e_1, -e_1, \vec{\tau})| = O(\beta^{3/2})\, e^{-1/2(m-O(1))\,|e_1|} \tag{3.13}$$

Suppose next that $\vec{\xi} = -\vec{\lambda}$, $|\vec{\xi}| = 1$. For instance take $\vec{\xi} = e_1 = \vec{\lambda}$. Then $\sum_{\vec{\tau}} |K(e_1, e_1, \vec{\tau})| = |K(e_1, e_1, -e_1)| + O(\beta^{3/2})\, e^{-1/2(m-O(1))\,|\vec{\xi}|}$ and using Lemma 3.2b (3.13) holds. Thus $\sum_{|\vec{\lambda}|=1} \sum_{\vec{\tau}} |K(\vec{\xi}, \vec{\lambda}, \vec{\tau})| \leqslant O(\beta)$ $e^{-1/2(m-O(1))\,|\vec{\xi}|}$ holds for $|\vec{\xi}| = 0, 1$.

Now we obtain a bound for $|\hat{M}(k^0)\,\hat{D}'_{0b}|_{\delta HS}$. We have

$$|\hat{M}(k^0)\,\hat{D}'_{0b}|^2_{\delta HS} \leqslant O(1) \sum_{\vec{\xi},\,\vec{\eta}} e^{2\delta(|\vec{\xi}|-|\vec{\eta}|)} \left(\sum_{\vec{\lambda}\neq\vec{0}} |K(\vec{\xi},\vec{\lambda},k^0)|^2\,(1-\delta(\vec{\eta}))\,|\vec{\eta}|^4 \right)$$

$$\leqslant O(1)\,\beta^{-2} e^{-2\delta} \sum_{\vec{\xi}} e^{2\delta\,|\vec{\xi}|} \left[\sum_{|\vec{\lambda}=1|} \sum_{\vec{\tau}} |K(\vec{\xi},\vec{\lambda},\vec{\tau})| \right]^2$$

$$+ O(1)\,e^{-2\delta} \sum_{\vec{\xi}} e^{-(m-O(1)-2\delta)\,|\vec{\xi}|}$$

The last term is $O(1)\,e^{-2\delta}$ and the first term is bounded by

$$O(1)\,\beta^{-2} e^{-2\delta} \left\{ \sum_{|\vec{\xi}|=0,\,1} O(\beta^2)\, e^{-m-O(1)-2\delta)\,|\vec{\xi}|} \right.$$

$$\left. + \sum_{|\vec{\xi}|\geqslant 2} O(\beta)\, e^{-(m-O(1)-2\delta)\,|\vec{\xi}|} \right\}$$

$$\leqslant O(1)\,e^{-2\delta}[1+O(1)\,e^{-1/2(m-O(1))}] \leqslant O(1)\,e^{-2\delta}$$

Thus $|\hat{M}(k^0)\,\hat{D}'_{0b}(k^0)|_{\delta HS} \leqslant O(\beta^{1/8})$ uniformly for $\chi \in (0, 2m)$ and the result follows.

We now give the proof of (b). The proof is the same as that for (a) with the exception that easily obtained bounds different from those of Lemma 3.2 are used for $\hat{D}'_{0a}$ and $\hat{D}'_{0b}$. These bounds are given by, with $\rho > 0$,

Lemma 3.6. For sufficiently small β and uniformly for $\chi \in [m_b + \delta\gamma, 2m)$ there exists a constant $c > 0$ such that

(a) $|\hat{D}'_{0a}(\vec{\xi},\vec{\eta},k^0)| \leqslant \rho^{-1}(1+c^{-1})$,

(b) $|\hat{D}'_{0b}(\vec{\xi},\vec{\eta},k^0)| \leqslant (1+c^{-1})\,O(1)[1-\delta(\vec{\xi})][1-\delta(\vec{\eta})](1+\beta^{-1}\,|\vec{\eta}|^2)$.

Proof. (a) From Eq. (3.12) we see that $|\hat{D}_0(\vec{\xi},\vec{\eta},\chi| \leqslant \hat{D}_0(\vec{0},\vec{0},\chi)$, $\hat{D}_0(\vec{0},\vec{0},\chi) > 0$ and monotone increasing. Thus, with $I \equiv [m_b+\delta\gamma, 2m)$, $\inf_{\chi\in I}(\rho\hat{D}_0(\vec{0},\vec{0},\chi)-1) = \rho\hat{D}_0(\vec{0},\vec{0},m_b+\delta\gamma)-1 \equiv c_1 \geqslant c$ and for the time being we assume $c>0$. We have

$$|\hat{D}'_{0a}(\vec{\xi},\vec{\eta},\chi)| \leqslant (\rho\hat{D}_0(\vec{0},\vec{0},\chi)-1)^{-1}\,\rho^{-1}(\rho\hat{D}_0(\vec{0},\vec{0},\chi)-1+1)$$

$$\leqslant \rho^{-1}(1+c^{-1})$$

(b) From the proof of Lemma 3.2b

$$\hat{D}'_{0b}(\vec{\xi},\vec{\eta},k^0) = -\rho(1-\rho D(\vec{0},\vec{0},k^0))^{-1}\,\hat{D}_0(\vec{0},\vec{0},k^0)\,B(\vec{\xi},\vec{\eta},k^0)$$

with $B(\vec{\xi}, \vec{\eta}, k^0) \equiv \hat{D}_0(\vec{0}, \vec{0}, k^0)^{-1} \Phi(\vec{\xi}, \vec{\eta}, k^0)$ and $|B(\vec{\xi}, \vec{\eta}, k^0)| \leqslant O(1)(1 + \beta^{-1} |\vec{\eta}|^2)$. Thus $|\hat{D}'_{0b}(\vec{\xi}, \vec{\eta}, \chi)| \leqslant (\rho \hat{D}_0(\vec{0}, \vec{0}, \chi) - 1 + 1)(\rho \hat{D}_0(\vec{0}, \vec{0}, \chi) - 1)^{-1} |B(\vec{\xi}, \vec{\eta}, \chi)| \leqslant (1 + c^{-1}) |B(\vec{\xi}, \vec{\eta}, \chi)|$. To complete the proof we show that $c > 0$. With $\chi = 2m - \varepsilon$, using the decomposition of $d\sigma_{\vec{p}}(\cdot)$ given by Eq. (3.11) in Eq. (3.12), we can write $\rho \hat{D}_0(\vec{0}, \vec{0}, k^0) \equiv F_1(\beta, \varepsilon) + F_2(\beta, \varepsilon) + F_3(\beta, \varepsilon)$ where $F_1(\beta, \varepsilon)$ is the contribution of the product of the one-particle contributions, etc. and $F_i(\varepsilon, \beta) > 0$, $i = 1, 2, 3$. Thus $\rho \hat{D}_0(\vec{0}, \vec{0}, k^0) > F_1(\beta, \varepsilon)$. $F_1(\beta, \varepsilon)$ is analyzed in Section II of ref. 6 where it is shown that $F_1(\beta, \varepsilon)$ is monotone decreasing in ε and jointly analytic in β and ε. Furthermore $F_1(0, \varepsilon) = \gamma/(1 - e^{-\varepsilon})$, $F_1(0, \varepsilon_0) = 1$ where $\varepsilon_0 = -\ln(1 - \gamma)$. Thus we have

$$c_1 > F_1(\beta, \varepsilon_0 - \delta\gamma + O(\beta)) - 1 = F_1(0, \varepsilon_0 - \delta\gamma) - F_1(0, \varepsilon_0) + O(\beta) \equiv c$$

As $F_1(0, \varepsilon_0 - \delta\gamma) - F_1(0, \varepsilon_0) = (1 - \gamma)(e^{\delta\gamma} - 1)(1 - (1 - \gamma)\, e^{\delta\gamma})^{-1} > 0$ it follows that $c > 0$ thus completing the proof.

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